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Summation of Logarithmic and Exponential Series.

By A. Chessin.

§1. Let us consider an infinite series, of which the general term is

$$F(n)\,x^n,$$

where $F(n)$ is a rational function of n. We can suppose that the degree of the numerator is less than that of the denominator, for if this were not the case we could break up the general term into two, of which one $E(n)\,x^n$ is the general term of a well known series,* $E(n)$ being an integral function of n, and the other one

$$\phi(n) = \frac{G(n)}{H(n)}\,x^n, \tag{1}$$

would satisfy the above condition, the degree of $G(n)$ being less than that of $H(n)$. We can also suppose the form (1) to be irreducible, for if $G(n)$ and $H(n)$ had common divisors we could cancel them. We will suppose, moreover, that μ being the degree of $H(n)$,

$$H(n) = C(n + a_1)(n + a_2)\,\ldots\ldots\,(n + a_\mu), \tag{2}$$

in which the $a_1, a_2 \ldots\ldots a_\mu$ are all different from each other, and can be any whole or fractional numbers, positive or negative. In the last case, however, it must be observed that the series may become divergent, as we shall see later.

Breaking up the fraction $\frac{G(n)}{H(n)}$ into partial ones, we shall have

$$\phi(n) = \sum_{e=1}^{e=\mu} \frac{A_e}{n + a_e}\,x^n. \tag{3}$$

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* See Chrystal's Algebra, Chap. XX, §13.

Now, let $a_e = \frac{b_e}{c_e}$, the fraction supposed to be irreducible, then

$$\phi(n) = \sum_{e=1}^{e=\mu} \frac{A_e c_e}{nc_e + b_e} x^n, \tag{4}$$

$$\sum_{n=0}^{n=\infty} \phi(n) = \sum_{e=1}^{e=\mu} A_e c_e \sum_{n=0}^{n=\infty} \frac{x^n}{c_e n + b_e}. \tag{5}$$

We see from these expressions that if one of the b_e is negative,

$$\frac{x^n}{c_e n + b_e}$$

will be infinite for a certain value of n, namely, for $n = -\frac{b_e}{c_e}$. But n being a whole number and $\frac{b_e}{c_e}$ irreducible, it can only happen when $c_e = 1$, i. e. when a_e is a whole number. And we see at once that the condition, sufficient and necessary for the elimination of infinite terms in (5) is, that *for every whole negative number* a_e *there should be another whole negative number* $a_{e'}$, *and that at the same time* $A_e x^{a_{e'}} + A_{e'} x^{a_e} = 0$. If none of the a_e are whole numbers, no infinite terms will occur in (5) and the series will be always convergent for $-1 < x < 1$.

If in the expression of the infinite series

$$\sum_{n=0}^{n=\infty} \frac{x^n}{c_e n + b_e}, \tag{6}$$

b_e is negative, we can transform it into one containing only positive terms in the denominator, and into a finite series. In fact we can write

$$\sum_{n=0}^{n=\infty} \frac{x^n}{c_e n + b_e} = \sum_{n=0}^{n=m-1} \frac{x^n}{c_e n + b_e} + \sum_{n=0}^{n=\infty} \frac{x^{n+m}}{c_e n + (c_e m + b_e)},$$

taking m so that $(c_e m + b) > 0$. And the first series on the right is a finite one; the second has the form

$$x^m \sum_{n=0}^{n=\infty} \frac{x^n}{c_e n + b'_e},$$

where $b'_e > 0$. Thus we can limit ourselves to the series (6) and suppose $b_e > 0$.

§2. To find the sum of the series

$$S=\sum_{0}^{\infty}\frac{x^n}{na+b} \tag{1}$$

we will first make the substitution

$$x=y^a, \tag{2}$$

so that

$$S=\sum_{0}^{\infty}\frac{y^{na}}{na+b}. \tag{1$'$}$$

Three cases are possible

1) $b<a$,
2) $b=ra$,
3) $b=ra+s \qquad s<a$.

The last one is the most general one. Cases 1) and 2) can be derived from it by making either $r=0$ or $s=0$. We will begin with case

3) $b=ra+s$,

$$\left.\begin{aligned} S=\sum_{0}^{\infty}\frac{y^{na}}{s+(n+r)a}&=y^{-ra}\sum_{n=r}^{n=\infty}\frac{y^{na}}{s+na} \\ &=-\sum_{n=0}^{n=r-1}\frac{y^{(n-r)a}}{s+na}+y^{-ra}\sum_{n=0}^{n=\infty}\frac{y^{na}}{s+na}. \end{aligned}\right\} \tag{3}$$

We know that

$$-\log(1-y)=\sum_{n=1}^{n=s-1}\frac{y^n}{n}+\sum_{n=0}^{n=\infty}\frac{y^{n+s}}{n+s}.$$

Let us put

$$\Theta(y)=-\log\,[(1-y)]^{y^{-s}}-\sum_{n=1}^{n=s-1}\frac{y^{n-s}}{n}=\sum_{n=0}^{n=\infty}\frac{y^n}{n+s}. \tag{4}$$

Now let $\rho_1, \rho_2 \dots . \rho_a$ be the roots of the equation

$$\rho^a=1.$$

We know that

$$\sum_{e=1}^{e=a}\rho_e^n\begin{cases} =0 \text{ when } m \neq \text{ a multiple of } a, \\ =a \text{ when } m=0 \text{ or } m=\text{ a multiple of } a. \end{cases}$$

Hence

$$\sum_{e=1}^{e=a}\Theta(\rho_e y)=\sum_{e=1}^{e=a}\sum_{n=0}^{n=\infty}\frac{(y\rho_e)^n}{n+s}=a\sum_{n=0}^{n=\infty}\frac{y^{na}}{na+s}. \tag{5}$$

But we also have

$$\left.\begin{aligned}\sum_{e=1}^{e=a}\Theta(\rho_e y)&=-\log\{(1-y\rho_1)^{(y\rho_1)^{-s}}(1-y\rho_2)^{(y\rho_2)^{-s}}\dots(1-y\rho_a)^{(y\rho\ )^{-s}}\}\\&\quad-\sum_{e=1}^{e=a}\sum_{n=1}^{n=s-1}\frac{(y\rho_e)^{n-s}}{n}\\&=-y^{-s}\log\{(1-y\rho_1)^{\rho_1^{-s}}(1-y\rho_2)^{\rho_2^{-s}}\dots(1-y\rho_a)^{\rho_a^{-s}}\}\end{aligned}\right\}. \tag{6}$$

This last expression can be easily transformed into

$$\left.-y^{-s}\sum_{e=1}^{e=a}\left\{\cos\left(\frac{2\pi es}{a}\right)\log\sqrt{1-2y\cos\frac{2\pi e}{a}+y^2}-\sin\frac{2\pi es}{a}\operatorname{tn}^{-1}\left(\frac{y\sin\frac{2\pi e}{a}}{1-y\cos\frac{2\pi e}{a}}\right)\right\}\right\}. \tag{7}$$

Thus,

$$\left.S=-\sum_{n=0}^{n=r-1}\frac{y^{(n-r)a}}{s+na}-\frac{y^{-b}}{a}\sum_{e=1}^{e=a}\left\{\cos\left(\frac{2\pi se}{a}\right)\log\sqrt{1-2y\cos\frac{2\pi e}{a}+y^2}-\sin\frac{2\pi se}{a}\operatorname{tn}^{-1}\left(\frac{y\sin\frac{2\pi e}{a}}{1-y\cos\frac{2\pi e}{a}}\right)\right\}\right\}. \tag{8}$$

1). When a is even, this formula can be written thus:

$$\left.\begin{aligned}S=&-\sum_{n=0}^{n=r-1}\frac{y^{(n-r)a}}{s+na}-\frac{1}{ay^b}\log(1-y)-\frac{(-1)^s}{ay^b}\log(1+y)\\&-\frac{1}{ay^b}\sum_{e=1}^{e=\frac{a}{2}-1}\cos\frac{2\pi se}{a}\log\left[1-2y\cos\frac{2\pi e}{a}+y^2\right]\\&+\frac{2}{ay^b}\sum_{e=1}^{e=\frac{a}{2}-1}\sin\frac{2\pi se}{a}\operatorname{tn}^{-1}\left(\frac{y\sin\frac{2\pi e}{a}}{1-y\cos\frac{2\pi e}{a}}\right)\end{aligned}\right\}. \tag{9$_1$}$$

2). When a is odd,

$$\left.\begin{aligned}S=&-\sum_{n=0}^{n=r-1}\frac{y^{(n-r)a}}{s+na}-\frac{1}{ay^b}\log(1-y)\\&-\frac{1}{ay^b}\sum_{e=1}^{e=\frac{a-1}{2}}\cos\frac{2\pi se}{a}\log\left[1-2y\cos\frac{2\pi e}{a}+y^2\right]\\&+\frac{2}{ay^b}\sum_{e=1}^{e=\frac{a-1}{2}}\sin\frac{2\pi se}{a}\,\mathrm{tn}^{-1}\left(\frac{y\sin\frac{2\pi e}{a}}{1-y\cos\frac{2\pi e}{a}}\right)\end{aligned}\right\}. \qquad (9)_2$$

Remark: When $y<0$ and $a=$ even number, the expression $(1)'$ shows that S does not change its value. It seems at a first glance as if $(9)_1$ did not satisfy this property, but it is very easy to see that the expression $(9)_1$ does not change after the substitution of $(-y)$ for y. For example, the term,

$$\frac{1}{ay^b}\sum_{e=1}^{e=\frac{a}{2}-1}\cos\frac{2\pi se}{a}\log\left[1-2y\cos\frac{2\pi e}{a}+y^2\right] \qquad (10)$$

after this substitution will change into

$$\frac{(-1)^s}{ay^b}\sum_{e=1}^{e=\frac{a}{2}-1}\cos\frac{2\pi se}{a}\log\left[1+2y\cos\frac{2\pi e}{a}+y^2\right]$$

$$=\frac{(-1)^s}{ay^b}\sum_{e_1=\frac{a}{2}-1}^{e_1=1}(-1)^s\cos\frac{2\pi se_1}{a}\log\left[1-2y\cos\frac{2\pi e_1}{a}+y^2\right],$$

and the last expression is obviously identical with (10).

In the special case, when

$$r=0,$$

we will have instead of formulæ $(9)_1$ and $(9)_2$ the following ones:

1). $a=$ even number,

$$\left.\begin{aligned}S=&-\frac{1}{ay^b}\log(1-y)-\frac{(-1)^b}{ay^b}\log(1+y)\\&-\frac{1}{ay^b}\sum_{e=1}^{e=\frac{a}{2}-1}\cos\frac{2\pi se}{a}\log\left[1-2y\cos\frac{2\pi e}{a}+y^2\right]\\&+\frac{2}{ay^b}\sum_{e=1}^{e=\frac{a}{2}-1}\sin\frac{2\pi se}{a}\,\mathrm{tn}^{-1}\left(\frac{y\sin\frac{2\pi e}{a}}{1-y\cos\frac{2\pi e}{a}}\right)\end{aligned}\right\}. \qquad (11)_1$$

2). $a =$ odd number,

$$\left.\begin{aligned} S = -\frac{1}{ay^b}\log(1-y) - \frac{1}{ay^b}\sum_{e=1}^{e=\frac{a-1}{2}}\cos\frac{2\pi se}{a}\log\left[1 - 2y\cos\frac{2\pi e}{a} + y^2\right] \\ + \frac{2}{ay^b}\sum_{e=1}^{e=\frac{a-1}{2}}\sin\frac{2\pi se}{a}\,\mathrm{tn}^{-1}\left(\frac{y\sin\frac{2\pi e}{a}}{1 - y\cos\frac{2\pi e}{a}}\right)\end{aligned}\right\}(11)_2$$

In the special case, when $r > 0$ and $s = 0$,

$$\sum_{e=1}^{e=a}\Theta(y\rho_e) = -\log(1-y^a),$$

and thus

$$S = -\frac{1}{ay^b}\log(1-y^a) - \frac{1}{ay^b}\sum_{n=1}^{n=r-1}\frac{y^{na}}{n} \tag{12}$$

Remark: We have used the substitution

$$x = y^a \tag{2}$$

to find the sum of the series (1). When $x < 0$ this substitution is not always justified. In fact if a is even and $x < 0$, there is no real-value of y that satisfies (2). To find the sum of the series (1) when $x < 0$ and a is an even number, we shall therefore make the substitution

$$x = -y^a, \tag*{(2)$_1$}$$

so that

$$S = \sum_{n=0}^{n=\infty}\frac{(-1)^n y^{na},}{na+b} \tag{13}$$

of which we will now find the expression.

§3. To find the sum (13) we proceed in the same way as we did to find the sum (1), §2. The only difference is that we shall have to consider the roots of the equation

$$\rho^a + 1 = 0$$

instead of those of the equation

$$\rho^a = 1.$$

Again,

$$\left.\begin{aligned} S_1=\sum_{n=0}^{n=\infty}\frac{(-1)^n y^{na}}{na+b} &= (-y^a)^{-r}\sum_{n=r}^{n=\infty}\frac{(-1)^n y^{na}}{s+na} \\ &= (-y^a)^{-r}\sum_{n=0}^{n=\infty}\frac{(-1)^n y^{na}}{s+na}-\sum_{n=0}^{n=r-1}\frac{(-1)^{n-r}y^{(n-r)a}}{s+na} \end{aligned}\right\}, \quad (1)$$

while $b=ra+s$. Introducing the function Θ again,

$$\Theta(y)=-\log(1-y)^{y^{-s}}-\sum_{n=1}^{n=s-1}\frac{y^{n-s}}{n}=\sum_{n=0}^{n=\infty}\frac{y^n}{n+s}.$$

We shall find in like manner

$$\sum_{e=1}^{e=a}\Theta(y\nu_e)=\sum_{e=1}^{e=a}\sum_{n=0}^{n=\infty}\frac{(y\nu_e)^n}{n+s}=a\sum_{n=0}^{n=\infty}\frac{(-1)^n y^{na}}{na+s}, \quad (2)$$

and also

$$\left.\begin{aligned} \sum_{e=1}^{e=a}\Theta(y\nu_e)=-y^{-s}\sum_{e=0}^{e=a-1}\Bigg\{ & \cos\frac{(2e+1)s\pi}{a}\log\sqrt{1-2y\cos\frac{(2e+1)\pi}{a}+y^2} \\ & -\sin\frac{(2e+1)s\pi}{a}\,\mathrm{tn}^{-1}\left(\frac{y\sin\frac{(2e+1)\pi}{a}}{1-y\cos\frac{(2e+1)\pi}{a}}\right)\Bigg\} \end{aligned}\right\}, \quad (3)$$

$\nu_1, \nu_2, \ldots . \nu_a$ being the roots of the equation $\rho^a+1=0$. Thus

$$\left.\begin{aligned} S_1=&-\sum_{n=0}^{n=r-1}\frac{(-1)^{n-r}y^{(n-r)a}}{s+na} \\ &-\frac{(-1)^r}{ay^b}\sum_{e=0}^{e=a-1}\Bigg\{\cos\frac{(2e+1)s\pi}{a}\log\sqrt{1-2y\cos\frac{(2e+1)\pi}{a}+y^2} \\ &-\sin\frac{(2e+1)s\pi}{a}\,\mathrm{tn}^{-1}\left(\frac{y\sin\frac{(2e+1)\pi}{a}}{1-y\cos\frac{(2e+1)\pi}{a}}\right)\Bigg\} \end{aligned}\right\}. \quad (4)$$

1). When a is even,

$$S_1 = \left. \begin{aligned} & -\sum_{n=0}^{n=r-1} \frac{(-1)^{n-r} y^{(n-r)a}}{s+na} \\ & -\frac{(-1)^r}{ay^b} \sum_{e=0}^{e=\frac{a}{2}-1} \left\{ \cos\frac{(2e+1)s\pi}{a} \log\left(1 - 2y\cos\frac{(2e+1)\pi}{a} + y^2\right) \right. \\ & \left. -2\sin\frac{(2e+1)s\pi}{a} \operatorname{tn}^{-1}\left(\frac{y\sin\frac{(2e+1)\pi}{a}}{1 - y\cos\frac{(2e+1)\pi}{a}} \right) \right\} \end{aligned} \right\} . \qquad (5)_1$$

2). When a is odd,

$$S_1 = \left. \begin{aligned} & -\sum_{n=0}^{n=r-1} \frac{(-1)^{n-r} y^{(n-r)a}}{s+na} - \frac{(-1)^b}{ay^b} \log(1+y) \\ & -\frac{(-1)^r}{ay^b} \sum_{e=0}^{e=\frac{a-3}{2}} \left\{ \cos\frac{(2e+1)s\pi}{a} \log\left(1 - 2y\cos\frac{(2e+1)\pi}{a} + y^2\right) \right. \\ & \left. -2\sin\frac{(2e+1)s\pi}{a} \operatorname{tn}^{-1}\left(\frac{y\sin\frac{(2e+1)\pi}{a}}{1 - y\cos\frac{(2e+1)\pi}{a}} \right) \right\} \end{aligned} \right\} . \qquad (5)_2$$

In the special case of $r = 0$ we have, when

1). a is even,

$$S_1 = \left. \begin{aligned} & -\frac{1}{ay^b} \sum_{e=0}^{e=\frac{a}{2}-1} \left\{ \cos\frac{(2e+1)b\pi}{a} \log\left(1 - 2y\cos\frac{(2e+1)\pi}{a} + y^2\right) \right. \\ & \left. -2\sin\frac{(2e+1)b\pi}{a} \operatorname{tn}^{-1}\left(\frac{y\sin\frac{(2e+1)\pi}{a}}{1 - y\cos\frac{(2e+1)\pi}{a}} \right) \right\} \end{aligned} \right\} . \qquad (6)_1$$

2). a is odd,

$$\left.\begin{aligned} S_1 = & -\frac{(-1)^b}{ay^b}\log(1+y) \\ & -\frac{1}{ay^b}\sum_{e=0}^{e=\frac{a-3}{2}}\left\{\cos\frac{(2e+1)b\pi}{a}\log\left(1-2y\cos\frac{(2e+1)\pi}{a}+y^2\right)\right. \\ & \left. -2\sin\frac{(2e+1)b\pi}{a}\,\mathrm{tn}^{-1}\left(\frac{y\sin\frac{(2e+1)\pi}{a}}{1-y\cos\frac{(2e+1)\pi}{a}}\right)\right\} \end{aligned}\right\}, \qquad (6)_2$$

and finally, when $s = 0$ and $r > 0$,

$$S_1 = -\frac{1}{ay^b}\log(1+y^a) - \frac{(-1)^r}{ay^b}\sum_{n=1}^{n=r-1}\frac{(-1)^n y^{na}.}{n} \qquad (7)$$

Thus we found the solution of our problem: to find the sum of the series

$$\sum_0^\infty \frac{x^n}{na+b}$$

for either positive or negative values of x. We shall now take up a few special cases.

§4. Let us consider the infinite series

$$\left.\begin{aligned} S_{p,\,q} & = \sum_{n=0}^{n=\infty}\frac{x^{na}}{[p+na][p+(n+1)a][p+(n+2)a]\ldots\ldots[p+(n+q)a]} \\ & = \sum_{n=0}^{n=\infty}\phi(p,\,q,\,n) \end{aligned}\right\}, \qquad (1)$$

where a, p, q are whole, positive numbers. The test of convergence is obvious. Instead of breaking up the general term of the series into partial fractions, as we did in the general case, we shall establish a formula of reduction which will lead us quicker to the final results.

Multiply (1) by x^{ma}, then the following relation between the functions $S_{p,\,q}$ will result:

$$x^{ma} S_{p,\,q} = S_{p-ma,\,q} - \sum_{n=0}^{n=m-1} \phi(p - ma,\, q,\, n). \tag{2}$$

Another relation is found thus:

$$\begin{aligned}
&\phi(p,\, q,\, n) - \phi(p,\, q,\, n+1) \\
&\qquad = (1 - x^a)\,\phi(p + a,\, q,\, n) + a(1 + q)\,\phi(p,\, q + 1,\, n), \\
&\phi(p,\, q,\, n+1) - \phi(p,\, q,\, n+2) \\
&\qquad = (1 - x^a)\,\phi(p + a,\, q,\, n+1) + a(1 + q)\,\phi(p,\, q + 1,\, n+1), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\phi(p,\, q,\, n+m) - \phi(p,\, q,\, n+m+1) \\
&\qquad = (1 - x^a)\,\phi(p + a,\, q,\, n+m) + a(1 + q)\,\phi(p,\, q + 1,\, n+m).
\end{aligned}$$

Adding all these formulæ, we get

$$\begin{aligned}
&\phi(p,\, q,\, n) - \phi(p,\, q,\, n+m+1) \\
&\qquad = (1 - x^a)\sum_{\mu=0}^{\mu=m} \phi(p + a,\, q,\, n+\mu) + a(1+q)\sum_{\mu=0}^{\mu=m} \phi(p,\, q+1,\, n+\mu).
\end{aligned}$$

But

$$\begin{aligned}
\phi(p + a,\, q,\, n + \mu) &= \phi(p + (n+1)a,\, q,\, \mu) . x^{na}, \\
\phi(p,\, q + 1,\, n + \mu) &= \phi(p + na,\, q + 1,\, \mu) . x^{na}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\phi(p,\, q,\, n) - \phi(p,\, q,\, n+m+1) \\
&\quad = (1 - x^a)x^{na}\sum_{\mu=0}^{\mu=m} \phi(p + (n+1)a,\, q,\, \mu) + a x^{na}(1+q)\sum_{\mu=0}^{\mu=m} \phi(p + na,\, q+1,\, \mu).
\end{aligned}$$

Now let m increase indefinitely. Then obviously, since we only deal with convergent series,

$$\lim_{m=\infty} \phi(p,\, q,\, n + m + 1) = 0,$$

and the above relation becomes

$$\phi(p,\, q,\, n) = x^{na}(1 - x^a)\, S_{p+(n+1)a,\, q} + a x^{na}(1 + q)\, S_{p+na,\, q+1}. \tag{3}$$

For our purposes we shall need only special cases of (2) and (3). Let in (2) m be $= 1$ and write $p + a$ for p; the formula then becomes

$$x^a S_{p+a,\,q} = S_{p,\,q} - \frac{1}{p\,(p+a)(p+2a)\,\ldots\ldots\,(p+qa)}\,. \tag{4}$$

In (3) let n be $= 0$. This will give

$$\frac{1}{p\,(p+a)(p+2a)\,\ldots\ldots\,(p+qa)} = (1 - x^a)\,S_{p+a,\,q} + a\,(1+q)\,S_{p,\,q+1}. \tag{5}$$

From (4) and (5) we get an interesting relation between three functions $S_{p,\,q}$; namely,

$$S_{p+a,\,q} = S_{p,\,q} - a\,(q+1)\,S_{p,\,q+1}. \tag{6}$$

From (4) and (6) we easily obtain the required formula of reduction,

$$S_{p,\,q} = \frac{1}{aqx^a}\left\{\frac{1}{p\,(p+a)(p+2a)\,\ldots\ldots\,[p+(q-1)\,a]} - (1-x^a)\,S_{p,\,q-1}\right\}. \tag{7}$$

I have substituted q for $q + 1$ and $q - 1$ for q in this formula. We have thus diminished the symbol (q) in $S_{p,\,q}$ by unity. Giving q all the values from 0 to q in formula (7), we shall, by successive reduction, obtain the formula

$$S_{p,\,q} = \frac{(-1)^q}{q\,!}\left(\frac{1-x^a}{ax^a}\right)^q S_{p,\,0} + \frac{1}{ax^a}\sum_{n=0}^{n=q-1} P_n\left(\frac{1-x^a}{ax^a}\right)^n, \tag{8}$$

$$P_n = \frac{(-1)^n}{q\,(q-1)(q-2)\ldots\ldots(q-n).p\,(p+a)(p+2a)\ldots\ldots[p+(q-n-1)\,a]}\,. \tag{9}$$

But $S_{p,\,0}$ is what we have called S before; that is,

$$S_{p,\,0} = \sum_{n=0}^{n=\infty}\frac{x^{na}}{p+na}\,, \qquad \text{(See §2, formula (1)')} \tag{10}$$

and we have found the expression of this sum (formulæ (8) and (9), §2); thus our problem is solved.

In the special case of $a = 1$, formulæ (8) and (9) reduce to

$$\left.\begin{aligned} S^0_{p,\,q} = {} & -\frac{(-1)^q(1-x)^q}{q\,!\;x^{p+q}}\left\{\log(1-x) + \sum_{n=1}^{n=p-1}\frac{x^n}{n}\right\} \\ & + \frac{1}{x}\sum_{n=0}^{n=q-1}\frac{(-1)^n}{q(q-1)\ldots\ldots(q-n).p(p+1)\ldots\ldots(p+q-n-1)}\left(\frac{1-x}{x}\right)^n \end{aligned}\right\}. \tag{11}$$

§5. Let us consider the series

$$S'_{p,\,q} = \sum_{n=0}^{n=\infty} \frac{(-1)^n x^{na}}{[p+na][p+(n+1)a]\ldots\ldots[p+(n+q)a]} = \sum_{n=0}^{n=\infty} \Psi(p,\,q,\,n). \qquad (1)$$

By a process similar to that used in §4 we shall find

$$S'_{p,\,q} = \frac{1}{q!}\left(\frac{1+x^a}{ax^a}\right)^q S'_{p,\,0} - \frac{1}{ax^a}\sum_{n=0}^{n=q-1} P'_n \left(\frac{1+x^a}{ax^a}\right)^n, \qquad (2)$$

where

$$P'_n = (-1)^n P_n. \qquad (3)$$

It is obvious that (2) could be directly received by substituting $(-x^a)$ for x^a in formula (8) of the preceding section.

Again,

$$S'_{p,\,0} = \sum_{n=0}^{n=\infty} \frac{(-1)^n x^{na}}{p+na}$$

is the series (13), §2, and we have given the expression of its sum in §3 (see formulæ (5), (6), (7)).

In the special case of $a = 1$, formula (2) reduces to

$$S^{0'}_{p,\,q} = -\frac{(-1)^p}{q!}\,\frac{(1+x)^p}{x^{p+q}}\left\{\log(1+x) + \sum_{n=1}^{n=p-1}\frac{(-1)^n x^n}{n}\right\}$$
$$-\frac{1}{x}\sum_{n=0}^{n=q-1}\frac{1}{q(q-1)\ldots\ldots(q-n).p(p+1)\ldots\ldots(p+q-n-1)}\left(\frac{1+x}{x}\right)^n.$$

§6. Let us now consider the series

$$S''_{p,\,q} = \sum_{n=0}^{n=\infty} \frac{x^n}{[p+na][p+(n+1)a]\ldots\ldots[p+(n+q)a]}. \qquad (1)$$

When $x > 0$, we reduce the case to that considered in §4 by the substitution

$$x = y^a.$$

When $x < 0$, we reduce it to the case considered in §5 by the substitution

$$-x = y^a.$$

Therefore we shall find the expressions for $S''_{p,\,q}$ if in the formula (8), §4, we substitute x for x^a, and in the formula (2), §5, substitute $-x$ for x^a. Thus,

$$S''_{p,\,q} = \frac{(-1)^q}{q\,!}\left(\frac{1-x}{ax}\right) S''_{p,\,0} + \frac{1}{ax}\sum_{n=0}^{n=q-1} P_n \left(\frac{1-x}{ax}\right)^n \tag{2}$$

for positive or negative values of x. But the expression of $S''_{p,\,0}$ will be different according to whether x is positive or negative. For

$$S''_{p,\,0} = \sum_{n=0}^{n=\infty} \frac{x^n}{p+na},$$

and we have seen that, to find the sum of this series, we must distinguish the two cases of $x \gtrless 0$. The expressions are given by the formulæ (9), §2, when $x > 0$, if in those formulæ we substitute $x^{1/a}$ for y; and by formulæ (5), §3, when $x < 0$, if we make the substitution $(-x)^{1/a}$ for y. Thus,

1). $x > 0$ and a is even:

$$\left.\begin{aligned} S''_{p,\,0} = {} & -\sum_{n=0}^{n=r-1} \frac{x^{n-r}}{s+na} - \frac{1}{ax^{p/a}} \log\left(1 - x^{1/a}\right) - \frac{(-1)^s}{ax^{p/a}} \log\left(1 + x^{1/a}\right) \\ & - \frac{1}{ax^{p/a}} \sum_{e=1}^{e=\frac{a}{2}-1} \cos\frac{2\pi se}{a} \log\left[1 - 2x^{1/a}\cos\frac{2\pi e}{a} + x^{2/a}\right] \\ & + \frac{2}{ax^{p/a}} \sum_{e=1}^{e=\frac{a}{2}-} \sin\frac{2\pi se}{a} \operatorname{tn}^{-1}\left(\frac{x^{1/a}\sin\dfrac{2\pi e}{a}}{1 - x^{1/a}\cos\dfrac{2\pi e}{a}}\right) \end{aligned}\right\}. \tag{3}$$

2). $x > 0$ and a is odd:

$$\left.\begin{aligned} S''_{p,\,0} = {} & -\sum_{n=0}^{n=r-1} \frac{x^{n-r}}{s+na} - \frac{1}{ax^{p/a}} \log\left(1 - x^{1/a}\right) \\ & - \frac{1}{ax^{p/a}} \sum_{e=1}^{e=\frac{a-1}{2}} \cos\frac{2\pi se}{a} \log\left[1 - 2x^{1/a}\cos\frac{2\pi e}{a} + x^{2/a}\right] \\ & + \frac{2}{ax^{p/a}} \sum_{e=1}^{e=\frac{a-1}{2}} \sin\frac{2\pi se}{a} \operatorname{tn}^{-1}\left(\frac{x^{1/a}\sin\dfrac{2\pi e}{a}}{1 - x^{1/a}\cos\dfrac{2\pi e}{a}}\right) \end{aligned}\right\}. \tag{4}$$

3). When $x < 0$ and a is even:

$$\left.\begin{aligned} S''_{p,0} = & -\sum_{n=0}^{n=r-1} \frac{x^{n-r}}{s+na} \\ & - \frac{(-1)^r}{a(-x)^{p/a}} \sum_{e=0}^{e=\frac{a}{2}-1} \left\{ \cos\frac{(2e+1)s\pi}{a} \log\left[1-2(-x)^{1/a}\cos\frac{(2e+1)\pi}{a}+(-x)^{2/a}\right] \right. \\ & \left. - 2\sin\frac{(2e+1)s\pi}{a}\,\mathrm{tn}^{-1}\left(\frac{(-x)^{1/a}\sin\frac{(2e+1)\pi}{a}}{1-(-x)^{1/a}\cos\frac{(2e+1)\pi}{a}}\right)\right\} \end{aligned}\right\}. \quad (5)$$

4). When $x < 0$ and a is odd:

$$\left.\begin{aligned} S''_{p,0} = & -\sum_{n=0}^{n=r-1} \frac{x^{n-r}}{s+na} - \frac{1}{ax^{p/a}}\log\left(1-x^{1/a}\right) \\ & - \frac{(-1)^s}{ax^{p/a}} \sum_{e=0}^{e=\frac{a-3}{2}} \cos\frac{(2e+1)s\pi}{a} \log\left[1+2x^{1/a}\cos\frac{(2e+1)\pi}{a}+x^{2/a}\right] \\ & + \frac{(-1)^s}{ax^{p/a}} \sum_{e=0}^{e=\frac{a-3}{2}} \sin\frac{(2e+1)s\pi}{a}\,\mathrm{tn}^{-1}\left(\frac{-x^{1/a}\sin\frac{(2e+1)\pi}{a}}{1+x^{1/a}\cos\frac{(2e+1)\pi}{a}}\right) \end{aligned}\right\}. \quad (6)$$

Formula (6) is identical with formula (4).

§7. Consider now the series

$$S^*_{p,q} = \sum_{n=0}^{n=\infty} \frac{x^n}{(p+n)(p+a+n)(p+2a+n)\ldots\ldots(p+qa+n)}; \quad (1)$$

it can be written in the following way:

$$\left.\begin{aligned} S^*_{p,q} & = \sum_{n=0}^{n=a-1}\sum_{m=0}^{m=\infty} \frac{x^{n+ma}}{[p+n+ma][p+n+(m+1)a]\ldots\ldots[p+n+(m+q)a]} \\ & = \sum_{n=0}^{n=a-1} x^n S_{p+n,q} \end{aligned}\right\}, \quad (2)$$

and as we found the expression for the sum $S_{p,\,q}$ in §4, we can consider the problem as solved.

§8. Let us now consider another type of series, the general term of which is

$$\frac{F(n)\,x^n}{(an+b)!},$$

where again $F(n)$ is a rational function of n; a and b are whole positive numbers. In the general case,

$$F(n)=E(n)+\frac{G(n)}{H(n)},$$

where $E(n)$ is an integral function of n and the degree of $G(n)$ is less than that of $H(n)$. We have thus to consider the two series:

$$1)\quad \sum_0^\infty \phi(n);\quad \phi(n)=\frac{E(n)\,x^n}{(an+b)!},$$

$$2)\quad \sum_0^\infty \psi(n);\quad \psi(n)=\frac{G(n)\,x^n}{H(n)(an+b)!}.$$

The summation of the first series can be reduced to the question of summing the series with the general term

$$\frac{E(n)\,x^n}{n!},$$

and this summation is known.* In fact, suppose we substitute $\frac{n-b}{a}$ for n in $E(n)$, and let

$$E\left(\frac{n-b}{a}\right)=E_1(n),$$

where $E_1(n)$ is also an integral function of n. We can find by known methods the sum of the series

$$\sum_0^\infty \frac{E_1(n)\,x^n}{n!}.$$

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* See Chrystal's Algebra, II, p. 209, on integro-exponential series.

Suppose it is equal to $\zeta_1(x)$. Then

$$\sum_0^\infty E\frac{\left(\frac{n-b}{a}\right)x^{n-b}}{n!} = x^{-b}\zeta_1(x). \tag{1}$$

But

$$\sum_0^\infty \frac{E\left(\frac{n-b}{a}\right)x^{n-b}}{n!} = \sum_0^{b-1}\frac{E\left(\frac{n-b}{a}\right)x^{n-b}}{n!} + \sum_0^\infty \frac{E\left(\frac{n}{a}\right)x^n}{(n+b)!},$$

and if we put

$$\vartheta_1(x) = x^{-b}\zeta_1(x) - \sum_0^{b-1}\frac{E_1(n)\,x^{n-b}}{n!}, \tag{2}$$

we shall have

$$\sum_0^\infty \frac{E\left(\frac{n}{a}\right)x^n}{(n+b)!} = \vartheta_1(x). \tag{3}$$

Now let again $\rho_1, \rho_2, \ldots . \rho_a$ be the roots of the equation

$$\rho^a = 1.$$

Then

$$\sum_{e=1}^{e=a}\sum_{n=0}^{n=\infty}\frac{E\left(\frac{n}{a}\right)(x\rho_e)^n}{(n+b)!} = a\sum_{n=0}^{n=\infty}\frac{E(n)\,x^{na}}{(na+b)!} = \sum_{e=1}^{e=a}\vartheta_1(x\rho_e),$$

or

$$\sum_{n=0}^{n=\infty}\frac{E(n)x^{na}}{(na+b)!} = \frac{1}{a}\sum_{e=1}^{e=a}\vartheta_1(x\rho_e) = \frac{1}{ax^b}\sum_{e=1}^{e=a}\frac{\zeta_1(x\rho_e)}{\rho_e^b}, \tag{4}$$

and as the expression on the right side is known, the function $\zeta(x)$ being known, the sum of the series (4) is found. And we need only substitute x for x^a to get the required sum of the series 1),

$$\sum_0^\infty \frac{E(n)\,x^n}{(an+b)!} = \frac{1}{ax^{b/a}}\sum_{e=1}^{e=a}\frac{\zeta_1(x^{1/a}\rho_e)}{\rho_e^b}. \tag{5}$$

Remark: We must observe again that we cannot make the substitution $x = y^a$ when $x < 0$ and a is an even number. In this case we shall again make the substitution $-x = y^a$, so that

$$\sum_0^\infty \frac{E(n)\,x^n}{(an+b)\,!} = \sum_0^\infty \frac{E(n)\,.(-1)^n y^{na}}{(an+b)\,!}\,, \tag{6}$$

and we shall introduce the roots $(\nu_1, \nu_2, \ldots . \nu_a)$ of the equation $\nu^a + 1 = 0$, instead of those of $\rho^a = 1$. Then, as it is easy to see,

$$\sum_0^\infty \frac{E(n)\,x^n}{(an+b)!} = \frac{1}{a\,(-x)^{b/a}} \sum_{e=1}^{e=a} \frac{\zeta\,[(-x)^{1/a}\nu_e]}{\nu_e^b}\,. \tag{7}$$

We have supposed a and b both to be whole positive numbers. It is obvious, however, that the solution is possible when b is negative, if only we observe that the expression $m\,!$ has a sense only when $m > 0$. Thus, if the general term were

$$\frac{E(n)\,x^n}{(an-b)\,!}$$

we would have to consider the values of $n > \frac{b}{a}$. Suppose n_1 is the first whole number $> \frac{b}{a}$, then we would consider the series

$$\sum_{n=n_1}^{n=\infty} \frac{E(n)\,x^n}{(an-b)\,!}\,. \tag{8}$$

Let us again put

$$E\left(\frac{n+b}{a}\right) = E_2(n)\,, \tag{9}$$

$$\zeta_2(x) = \sum_0^\infty \frac{E_2(n)\,x^n}{n\,!}\,, \tag{10}$$

$$\vartheta_2(x) = x^b \zeta_2(x) + \sum_{n=n_1}^{n=n_1+b-1} \frac{E\left(\frac{n}{a}\right) x^n}{(n-b)\,!}\,. \tag{11}$$

In the expression of the last sum only those terms are supposed to be taken for which the expression $(n-b)\,!$ has a sense, i. e. for which $n - b > 0$. Then

$$\sum_{n=n_1}^{n=\infty} \frac{E\left(\frac{n}{a}\right) x^n}{(n-b)\,!} = \vartheta_2(x)\,.$$

Here also only terms, for which $n - b > 0$, are to be taken. Operating as before we find

$$\sum_{n=n_1}^{n=\infty} \frac{E(n)\, x^{na}}{(na-b)!} = \frac{1}{a} \sum_{e=1}^{e=a} \vartheta_2(x\rho_e) = \frac{x^b}{a} \sum_{e=1}^{e=a} \rho_e^b \zeta_2(x\rho_e), \tag{12}$$

or

$$\sum_{n=n_1}^{n=\infty} \frac{E(n)\, x^{n}}{(na-b)!} = \frac{x^{b/a}}{a} \sum_{e=1}^{e=a} \rho_e^b \zeta_2(x^{1/a}\rho_e) \tag{13}$$

when $x > 0$ and

$$\sum_{n=n_1}^{n=\infty} \frac{E(n)\, x^{n}}{(na-b)!} = \frac{(-x)^{b/a}}{a} \sum_{e=1}^{e=a} \nu_e^b \zeta_2[(-x)^{1/a}\nu_e]$$

when $x < 0$.

§9. Let us now take the second part of our series:

$$\sum_0^{\infty} \psi(n); \quad \psi(n) = \frac{G(n)\, x^n}{H(n)\,(an+b)!}.$$

We will again suppose that $\frac{G(n)}{H(n)}$ is in irreducible form and that

$$H(n) = C(n+\alpha_1)(n+\alpha_2) \ldots . (n+\alpha_\mu),$$

in which expression the $\alpha_1 \ldots . \alpha_\mu$ are all different from each other and can be any whole or fractional numbers, positive or negative, with the same restriction in the last case as in §1. The whole numbers a and b are again positive. Breaking up the function into partial fractions, we shall obtain expressions of the form

$$\frac{A_e\, x^n}{(n+\alpha_e)(an+b)!}, \tag{1}$$

and we shall have

$$\sum_0^{\infty} \psi(n) = \sum_{n=0}^{n=\infty} \sum_{e=1}^{e=\mu} \frac{A_e x^n}{(n+\alpha_e)(an+b)!}. \tag{2}$$

Let again $\alpha_e = \frac{\beta_e}{\gamma_e}$, and let the fraction be irreducible, then

$$\sum_0^\infty \psi(n) = \sum_{e=1}^{e=\mu} A_e \gamma_e \sum_{n=0}^{n=\infty} \frac{x^n}{(\gamma_e n + \beta_e)(an+b)!}. \tag{3}$$

We see again that only when a number α_e is a *whole negative number*, infinite terms will occur in (3). And the condition necessary and sufficient for the elimination of such terms is that *for every such* α_e *there should be another whole negative number* $\alpha_{e'}$, *and that at the same time*

$$\frac{A_e x^{-\alpha^e}}{(-a\alpha_e + b)!} + \frac{A_{e'} x^{-\alpha_{e'}}}{(-a\alpha_{e'} + b)!} = 0.$$

If these conditions subsist, the series will be convergent for all positive and negative values of x (except $x = \infty$).

If in (3) any of the β_e are negative, we can reduce the expression to one in which the corresponding number will be positive, by operating just in the same way as in §1. Thus all is reduced to find the sum of the series

$$\sum_0^\infty \frac{x^n}{(\alpha n + \beta)(an+b)!}, \tag{4}$$

where α, β, a, b are all positive whole numbers.

§10. From the expression of the development of e^x into a series we deduce the formula

$$x^{-\lambda b} e^{x^\lambda} = \sum_{n=0}^{n=b-1} \frac{x^{\lambda(n-b)}}{n!} + \sum_{n=0}^{n=\infty} \frac{x^{\lambda n}}{(n+b)!}, \tag{1}$$

in which $\lambda = \frac{\alpha}{a}$. Multiply (1) by $x^{\beta-1}$ and integrate between the limits 0 and x:

$$\int_0^x e^{x^\lambda} x^{\beta - \lambda b - 1} dx = \sum_{n=0}^{n=b-1} \frac{x^{\lambda(n-b)+\beta}}{n!\,[\lambda(n-b)+\beta]} + \sum_{n=0}^{n=\infty} \frac{x^{\lambda n + \beta}}{(n+b)!\,(\lambda n + \beta)}.$$

Let us introduce the function

$$\Psi(x) = x^{-\beta} \int_0^x e^{x^\lambda} x^{\beta - \lambda b - 1} dx - \sum_{n=0}^{n=b-1} \frac{x^{\lambda(n-b)}}{n!\,[\lambda(n-b)+\beta]}. \tag{2}$$

Then
$$\sum_{n=0}^{n=\infty} \frac{x^{\lambda n}}{(n+b)!\,(\lambda n+\beta)} = \Psi(x). \tag{3}$$

Let again $\rho_1, \rho_2, \ldots \rho_a$ be the roots of the equation
$$\rho^a = 1.$$

Then by the same device as before we shall find
$$\sum_{e=1}^{e=a} \Psi(x\rho_e^{1/\lambda}) = \sum_{n=0}^{n=\infty}\sum_{e=1}^{e=a} \frac{\rho_e^n x^{\lambda n}}{(n+b)!\,(\lambda n+\beta)} = a\sum_{n=0}^{n=\infty} \frac{x^{an}}{(an+b)!\,(an+\beta)}. \tag{4}$$

In this formula substitute $x^{1/a}$ for x (when $x > 0$). Then
$$\sum_{0}^{\infty} \frac{x^n}{(an+b)!\,(\alpha n+\beta)} = \frac{1}{a}\sum_{e=1}^{e=a} \Psi\,[x^{1/a}\rho_e^{a/\alpha}] = \frac{1}{a}\sum_{e=1}^{e=a} \Psi\,[x^{1/a}\rho_e'], \tag{5}$$
where
$$\rho_e' = \cos\frac{2\pi e}{\alpha} + i\sin\frac{2\pi e}{\alpha}; \tag{6}$$

that is to say, the ρ_e' are roots of the equation $\rho^\alpha = 1$; but as $a \neq \alpha$, we shall not generally have a complete set of these roots in the expression on the right of (5). In formula (2) the sum on the right will vanish when $b < a$ after the substitution $x\rho_e^{1/\lambda}$ for x; when $b > 0$, those terms for which $b - n =$ a multiple of a will remain on account of the property of the roots of unity.

For $x < 0$ we shall find the formula
$$\sum_{0}^{\infty} \frac{x^n}{(an+b)!\,(\alpha n+\beta)} = \frac{1}{a}\sum_{e=1}^{e=a} \Psi\,[(-x)^{1/a}\nu_e^{a/\alpha}] = \frac{1}{a}\sum_{e=1}^{e=a} \Psi\,[(-x)^{1/a}\nu_e'], \tag{7}$$
where
$$\nu_e' = \cos\frac{(2e+1)\pi}{\alpha} + i\sin\frac{(2e+1)\pi}{\alpha}; \tag{8}$$

that is to say, the ν_e' are roots of the equation $\nu^\alpha + 1 = 0$. Thus we can consider our problem as solved.

Remark: We made the supposition $b > 0$ throughout the last analysis. But the results are obviously valid for $b < 0$ if we only remember that in this case the summation must be done from $n = n_1$ to $n = \infty$, where n_1 is the first whole number greater than $-\frac{b}{a}$.

§11. The conditions necessary and sufficient for the integrability of the expression (2), §10, are obviously the following ones:

$$\left.\begin{array}{ll} 1) & \beta = \varkappa\lambda = \varkappa\,\dfrac{\alpha}{a}\,, \\ 2) & \varkappa \geqq b + 1\,. \end{array}\right\} \tag{1}$$

Then

$$\Psi(x) = \frac{x^{-\kappa\lambda}}{\lambda}\int_0^{x^\lambda} e^y y^{\kappa-b-1}dy - \sum_{n=0}^{n=b-1}\frac{x^{\lambda(n-b)}}{n!\,[\lambda(n-b)+\beta]}\,. \tag{2}$$

Suppose

$$\varkappa - (b+1) = m\,. \tag{3}$$

Then

$$\left.\begin{array}{l} \displaystyle\int_0^{x^\lambda} e^y y^m dy = e^{x^\lambda}\left[x^{\lambda m} - m x^{\lambda(m-1)} + m(m-1)\,x^{\lambda(m-2)}\right. \\ \qquad\qquad\qquad \left. - \ldots\ldots \pm m(m-1)\ldots\ldots 2.1\right] \mp (m!) \end{array}\right\}. \tag{4}$$

But if the conditions (1) subsist, we shall have

$$\frac{1}{\alpha n + \beta} = \frac{a}{\alpha(na + \varkappa)}$$

and

$$\sum_0^\infty \frac{x^n}{(an+b)!\,(\alpha n + \beta)} = \frac{a}{\alpha}\sum_0^\infty \frac{x^n}{(an+b)!\,(an+\varkappa)}\,;$$

that is to say, we can suppose in formulæ (5) and (7), §10,

$$\alpha = a,$$
$$\beta = \varkappa,$$

if only we multiply the formulæ by $\frac{a}{\alpha}$, hence

$$\sum_0^\infty \frac{x^n}{(an+b)!\,(\alpha n+\beta)} = \frac{1}{\alpha}\sum_{e=1}^{e=a}\Psi\left[x^{1/a}\rho_e\right], \tag{5}$$

when $x > 0$, and

$$\sum_0^\infty \frac{x^n}{(an+b)!\,(\alpha n+\beta)} = \frac{1}{\alpha}\sum_{e=1}^{e=a}\Psi\left[(-x)^{1/a}\nu_e\right], \tag{6}$$

when $x < 0$. Moreover, we have then

$$\Psi(x) = x^{-\kappa}\int_0^x e^x x^m dx - \sum_{n=0}^{n=b-1} \frac{x^{n-b}}{n!\,(n-b+\kappa)}, \tag{7}$$

$$\left.\begin{aligned}\int_0^x e^x x^m dx = e^x\,[x^m - m x^{m-1} + m(m-1)\,x^{m-2}&\\ - \ldots . \pm m(m-1) \ldots . 2.1] \mp (m!),&\end{aligned}\right\} \tag{8}$$

and from these formulæ it will be easy to deduce

$$\sum_{e=1}^{e=a} \Psi\,[x^{1/a}\rho_e] \text{ and } \sum_{e=1}^{e=a} \Psi\,[(-x)^{1/a}\nu_e].$$

Interesting applications can be made, but we will not go into further details.